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STATISTICAL CHARACTERISTICS OF A PASSIVE ADMIXTURE IN A HOMOGENEOUS ISOTROPIC TURBULENCE FIELD

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1. It is well known that in studying different physical phenomena, in particular in order to understand mixing processes, it is necessary to know the spectral characteristics of the passive admixture located in a developed turbulence field [1]. Information on the statistical properties of the corresponding scalar fields (concentration, temperature, moisture content and so on) is important in analyzing the propagation and scattering of acoustical, optical, and radio waves in a turbulent medium [2].

In this paper, we study the spectral structure of a passive admixture with the help of a regular procedure, based on Wyld's diagrammatic technique [3]. Using improved approximations of direct interactions, we find the spectrum of the passive impurity in the inertial-convective interval, obtained previously from dimensional considerations [4, 5] and semiempirical theories, which are reviewed in [6, 7]. The flow direction of the passive admixture is determined from the scale spectrum. The asymptotic behavior of the spectrum is studied in the viscodiffusion interval of wave numbers. For generality of the presentation, the spectral characteristics are analyzed in a space with arbitrary dimensionality d.

2. In order to describe the passive admixture in the homogeneous isotropic turbulence field, we shall examine the Navier–Stokes equations, the equation of continuity, and the diffusion equation, which in the **k** representation have the form

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)v_{\mathbf{k}}^{\alpha}=-\frac{i}{2}P_{\mathbf{k}}^{\alpha\beta\gamma}\int v_{\mathbf{k}_1}^{*\beta}v_{\mathbf{k}_2}^{*\gamma}\delta^{(d)}(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)\,d^{(d)}\mathbf{k}_1d^{(d)}\mathbf{k}_2,$$

$$k_\alpha v_{\mathbf{k}}^{\alpha}=0,$$

$$\left(\frac{\partial}{\partial t}+\chi k^2\right)\vartheta_{\mathbf{k}}=-ik_\alpha\int v_{\mathbf{k}_1}^{*\alpha}\vartheta_{\mathbf{k}_2}^{*}\delta^{(d)}(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)\,d^{(d)}\mathbf{k}_1d^{(d)}\mathbf{k}_2,$$

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where $P^{\alpha\beta\gamma}_{\mathbf{k}} = k_\beta \Delta^{\alpha\gamma}_{\mathbf{k}} + k_\gamma \Delta^{\alpha\beta}_{\mathbf{k}}$; $\Delta^{\alpha\beta}_{\mathbf{k}} = \delta_{\alpha\beta} - k_\alpha k_\beta / k^2$; $\delta_{\alpha\beta}$ is the Kronecker symbol; $v^\alpha_{\mathbf{k}}$ and $\vartheta_{\mathbf{k}}$ are the dimensional Fourier transformations of the velocity field and of the concentration of the passive admixture:

$$v^\alpha_{\mathbf{k}}(t) = \int \frac{d^{(d)}\mathbf{r}}{(2\pi)^d} v^\alpha(\mathbf{r}, t)\, e^{-i\mathbf{k}\mathbf{r}}, \quad \vartheta_{\mathbf{k}}(t) = \int \frac{d^{(d)}\mathbf{r}}{(2\pi)^d} \vartheta(\mathbf{r}, t)\, e^{-i\mathbf{k}\mathbf{r}};$$

ν is the kinematic coefficient of molecular viscosity and χ is the coefficient of molecular diffusion (thermal conductivity).

In what follows, for definiteness we shall assume that $\vartheta(\mathbf{r}, t)$ is the temperature imparted to the fluid, but not having a significant effect on the dynamics of turbulence.

We shall go over to a statistical description of the isotropic velocity and temperature fields. For this, we shall use Wyld's diagrammatic technique [3], which makes use of two characteristics of each nonlinear field: the pair correlation function and Green's function. We shall introduce the diagrams of these quantities in the $\mathbf{k}$ representation (see Fig. 1):

$$\langle v^\alpha_{\mathbf{k}\omega} v^{*\beta}_{\mathbf{k}'\omega'} \rangle = F^{\alpha\beta}_q \delta^{(d+1)}(q - q'), \quad \langle \vartheta_{\mathbf{k}\omega} \vartheta^*_{\mathbf{k}'\omega'} \rangle = I_q \delta^{(d+1)}(q - q'), \tag{2.1}$$
$$\left\langle \frac{\delta v^\alpha_{\mathbf{k}\omega}}{\delta f^{*\beta}_{\mathbf{k}'\omega'}} \right\rangle = G^{\alpha\beta}_q \delta^{(d+1)}(q - q'), \quad \left\langle \frac{\delta \vartheta_{\mathbf{k}\omega}}{\delta f^{*(\vartheta)}_{\mathbf{k}'\omega'}} \right\rangle = g_q \delta^{(d+1)}(q - q'),$$
$$q = (\mathbf{k}, \omega), \quad \delta^{(d+1)}(q) \equiv \delta(\omega)\, \delta^{(d)}(\mathbf{k}).$$

Due to the isotropy of turbulence, the spectral tensors $F^{\alpha\beta}_q$ and $G^{\alpha\beta}_q$ can be represented in the form

$$F^{\alpha\beta}_{\mathbf{k}\omega} = F_{\mathbf{k}\omega} \Delta^{\alpha\beta}_{\mathbf{k}}, \; G^{\alpha\beta}_{\mathbf{k}\omega} = G_{\mathbf{k}\omega} \Delta^{\alpha\beta}_{\mathbf{k}}.$$

The auxiliary quantities $G_{\mathbf{k}\omega}$ and $g_{\mathbf{k}\omega}$ describe the reaction of the turbulent velocity and temperature fields to external perturbations $f_{\mathbf{k}\omega}$ and $f^{(\vartheta)}_{\mathbf{k}\omega}$, introduced on the right sides of the Navier–Stokes equations and the heat conduction equation.

The spectral characteristics of (2.1) satisfy the Dyson equation

$$F_{\mathbf{k}\omega} = |G_{\mathbf{k}\omega}|^2 \Phi_{\mathbf{k}\omega}, \quad G_{\mathbf{k}\omega} = (\omega + i\nu k^2 - \Sigma_{\mathbf{k}\omega})^{-1}, \tag{2.2a}$$
$$I_{\mathbf{k}\omega} = |g_{\mathbf{k}\omega}|^2 \varphi_{\mathbf{k}\omega}, \quad g_{\mathbf{k}\omega} = (\omega + i\chi k^2 - \sigma_{\mathbf{k}\omega})^{-1}. \tag{2.2b}$$

The first diagrams for Φ_q, Σ_q and φ_q, σ_q are presented in Fig. 1, where the triangle indicates the vertex $\Gamma^{\alpha\beta\gamma}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} \delta^{(d+1)}(q + q_1 + q_2)$ $\left(\Gamma^{\alpha\beta\gamma}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} = \left[\Delta^{\alpha\alpha_1}_{\mathbf{k}} k_{\beta_1} + \Delta^{\alpha\beta_1}_{\mathbf{k}} k_{\alpha_1}\right] \Delta^{\alpha_1\beta}_{\mathbf{k}_1} \Delta^{\beta_1\gamma}_{\mathbf{k}_2}\right)$, while the circle indicates the quantity $k_\alpha \delta^{(d+1)}(q + q_1 + q_2)$.

The analysis of the starting diagrammatic series (see Fig. 1) encounters well-known difficulties, related to the divergence of integrals in the region of small $\mathbf{k}$. Physically, these divergences are due to the effect of transport of small-scale energy pulsations containing vortices. Part of the most divergent diagrams for hydrodynamic turbulence, which describe the transport interactions, was summed in [8]. In this case, the improved equations admit in the direct interaction approximation (which corresponds to the inclusion of second-order diagrams with respect to the vertices $\Gamma^{\alpha\beta\gamma}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$) in the three-dimensional [9] and two-dimensional [10] cases exact solutions in the form of Kolmogorov spectra. We shall present

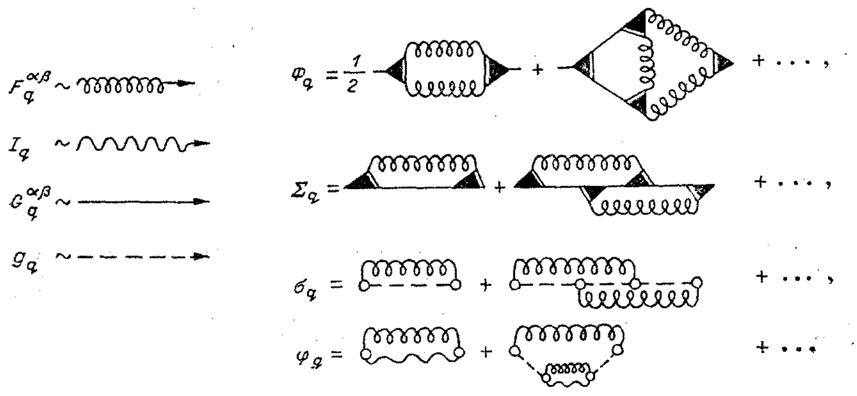

Fig. 1.

the corresponding values of the indices for the pair correlation function of the velocity field, which we will need in analyzing the spectra of temperature pulsations:

$$\widetilde{F}_{\mathbf{k}\omega} = k^{-(\alpha+\beta)} f(\omega/k^{\beta})$$

(the function $\tilde{F}_q$ is related to the starting function F_q by the relation $F_{\mathbf{k}\omega} = \langle \widetilde{F}_{\mathbf{k}\omega-\mathbf{k}\mathbf{v}} \rangle_{\mathbf{v}}$, where $\langle \cdots \rangle_{\mathbf{v}}$ indicates averaging over the random velocity field at an arbitrary point ($\mathbf{r}$, t) with the help of Wyld's procedure).

In the three-dimensional case (d = 3), $\alpha = 11/3$, $\beta = 2/3$; in the two-dimensional case (d = 2), two solutions exist with $\alpha = 8/3$, $\beta = 2/3$ and $\alpha = 4$, $\beta = 0$, respectively and, in addition, the first solution corresponds to a spectrum with constant energy flux ε_0, while the second corresponds to a spectrum with constant enstrophy flux ε_2.

When transport is included, certain subtractions appear in diagrams for σ_q and φ_q and, in this case, within the direct interaction model, the corresponding quantities have the form

$$\tilde{\sigma}_q = \int k_\alpha k_{2\beta} \Delta^{\alpha\beta}_{\mathbf{k}_1} \tilde{g}_{q_2} \widetilde{F}_{q_1} \left[\delta^{(d+1)}(q + q_1 + q_2) - \delta^{(d+1)}(q + q_1)\right] dq_1 dq_2, \tag{2.3}$$

$$\tilde{\varphi}_q = \int k_\alpha k_\beta \Delta^{\alpha\beta}_{\mathbf{k}_1} \widetilde{I}_{q_2} \widetilde{F}_{q_1} \left[\delta^{(d+1)}(q + q_1 + q_2) - \delta^{(d+1)}(q + q_1)\right] dq_1 dq_2,$$

$$g_{\mathbf{k}\omega} = \langle \tilde{g}_{\mathbf{k}\omega-\mathbf{k}\mathbf{v}} \rangle_{\mathbf{v}}, \quad I_{\mathbf{k}\omega} = \langle \widetilde{I}_{\mathbf{k}\omega-\mathbf{k}\mathbf{v}} \rangle_{\mathbf{v}}.$$

The spectral functions $\tilde{g}_{\mathbf{k}\omega}$ and $\widetilde{I}_{\mathbf{k}\omega}$, in their turn, likewise satisfy the Dyson equations:

$$\widetilde{I}_{\mathbf{k}\omega} = |\tilde{g}_{\mathbf{k}\omega}|^2 \tilde{\varphi}_{\mathbf{k}\omega}, \quad \tilde{g}_{\mathbf{k}\omega} = (\omega + i\chi k^2 - \tilde{\sigma}_{\mathbf{k}\omega})^{-1}. \tag{2.4}$$

3. We shall first examine the statistical characteristics of the temperature field in the inertial-convective interval $\min(L, L_\vartheta) = L_0 \gg k^{-1} \gg \eta_0 = \max(\eta, \eta\ )$, where L, L are the characteristic scales of the energy containing part of the spectrum of the velocity and temperature fields; $\eta = (\nu^3/\varepsilon)^{1/4}$, $\eta\ = (\chi^3/\varepsilon)^{1/4}$ are the internal Kolmogorov scales of the corresponding fields.* Internal friction and molecular thermal conductivity can be neglected for perturbations with scales in these intervals.

We first note that the system of equations (2.4) admits a thermodynamically equilibrium solution $\tilde{I}_q = (T/\pi) \operatorname{Im} \tilde{g}_q$, which results in exceptional situations, when it is possible to neglect sources and sinks of turbulence. The system (2.4) also contains nonequilibrium flow distributions, which (with $\nu = \chi = 0$) we shall seek in a scale-invariant form:

$$\widetilde{I}_q = k^{-(t+p)} r(\omega/k^s), \quad \tilde{g}_q = k^{-s} h(\omega/k^s). \tag{3.1}$$

The solution with the flow (3.1), for example, under oceanographical conditions according to Phillips [1], corresponds to the following physical picture. The random entrainment of cold water from below creates a source of temperature fluctuations in the upper layer of the ocean and, in this case, the characteristic initial scale of these pulsations corresponds to the scale of liquid vortices responsible for the entrainment. The cascade process of fragmentation of the vortices of the velocity field into smaller and smaller perturbations (with d = 3) will lead simultaneously to fragmentation of inhomogeneities of the temperature field until molecular heat conduction becomes important, which leads to an equalization of temperature at nearby points. Thus, as a result of entrainment and the subsequent cascade fragmentation process due to turbulent mixing, there arises a flow of the measure of temperature inhomogeneities over the spectrum of scales.

The expression for the index s can be obtained directly from the Dyson equations (2.4)

$$s = 2 + d - (\alpha + \beta). \tag{3.2}$$

In order to find the index p, we shall form in a standard manner [11] the combination

$$\widetilde{I}_q = \operatorname{Im}(\tilde{\varphi}_q \tilde{g}_q^* + \widetilde{I}_q \tilde{\sigma}_q) = 0, \tag{3.3}$$

equivalent to the Dyson equation for $\tilde{I}_q$.

Integrating (3.3) with respect to ω and performing next the conformal transformation [9]

$$k = k''(k/k''), \quad k_1 = k'(k/k''), \quad k_2 = k(k/k''),$$

$$\omega = \omega''(k/k'')^s, \quad \omega_1 = \omega'(k/k'')^s, \quad \omega_2 = \omega(k/k'')^s,$$

*In the two-dimensional case, the scales η and η_ϑ play the role of $\eta_2 = (\nu/\varepsilon_2^{1/3})^{1/2}$ and $\eta_{2\vartheta} = (\chi/\varepsilon_2^{1/3})^{1/2}$.

we obtain the equation

$$0=\tilde{l}_{\mathbf{k}}=\frac{\mathrm{Im}}{2}\int d\omega dq_1dq_2\delta^{(d+1)}(q+q_1+q_2)\left(k_\beta\tilde{g}_q\tilde{I}_q^{-1}+k_{2\beta}\tilde{g}_{q_2}\tilde{I}_{q_2}^{-1}\right)\left[k_\alpha+(k/k_2)^x k_{2\alpha}\right]\Delta^{\alpha\beta}_{\mathbf{k}_1}\tilde{F}_{q_1}\tilde{I}_q\tilde{I}_{q_2}, \tag{3.4}$$

where $x = 2 + 2d - p - (\alpha + \beta)$.

Together with the equilibrium solution $\tilde{I}_q = (T/\pi)\,\mathrm{Im}\,\tilde{g}_q$ from (3.4), it is evident that there is also another solution with $x = 0$, for which the expression in square brackets vanishes due to the obvious identity

$$(k_\alpha+k_{2\alpha})\,\Delta^{\alpha\beta}_{\mathbf{k}_1}\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)=0,$$

reflecting the presence of the integral of motion $\int|\vartheta_{\mathbf{k}}|^2d^{(d)}\mathbf{k}$ of the starting system of equations. Thus, for the index p, we obtain the relation

$$p=2+2d-(\alpha+\beta). \tag{3.5}$$

From (3.2) and (3.5), we also obtain, for the three-dimensional case using the Kolmogorov values of α and β, Kolmogorov expressions for p and s: $p = 11/3$, $s = 2/3$.

For $d = 2$, we have two spectra with $p = 8/3$, $s = 2/3$ and $p = 2$, $s = 0$, respectively.

A simple analysis, analogous to that performed in [12, 10], shows that the integrals in (2.3) and (3.3) over the distributions found converge at the lower and upper limits and, thus, the spectra obtained are local spectra.

4. Let us examine the direction of flow of the temperature inhomogeneities over the spectrum of scales. For this, we shall write the equation of balance for the spectral intensity of temperature pulsations

$$\frac{\partial I_{\mathbf{k}}}{\partial t}=-2\,\mathrm{Im}\,k_\alpha\int J^\alpha_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}\delta^{(d)}(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)\,d^{(d)}\mathbf{k}_1d^{(d)}\mathbf{k}_2, \tag{4.1}$$

where $J^\alpha_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}=\left\langle v^\alpha_{\mathbf{k}_1}\vartheta_{\mathbf{k}_2}\vartheta_{\mathbf{k}}\right\rangle$.

It has the form of a continuity equation, so that the right side can be represented as a divergence of the flux of the measure of inhomogeneity in the temperature field:

$$\varepsilon_T=-(1/2)\,S(d)\int_0^k k^{d-1}dk\,2\,\mathrm{Im}\,k_\alpha\int J^\alpha_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}\delta^{(d)}(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)\,d^{(d)}\mathbf{k}_1d^{(d)}\mathbf{k}_2. \tag{4.2}$$

Here, $S(d) = 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of a unit sphere in d-dimensional space and $\Gamma(\cdot)$ is the gamma function.

Let us express the triple correlation function $J^\alpha_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$ as a series in powers of $\tilde{g}_q$, $\tilde{I}_q$, and $\tilde{F}_q$, and in this case in the direct-interaction approximation for the stationary case, we have*

$$J^\alpha_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}=\left\langle\int\tilde{J}^\alpha_{qq_1q_2}\delta(\omega+\omega_1+\omega_2)\,d\omega d\omega_1d\omega_2\right\rangle_{\mathbf{v}}\equiv\int\tilde{J}^\alpha_{qq_1q_2}\delta(\omega+\omega_1+\omega_2)\,d\omega d\omega_1d\omega_2, \tag{4.3}$$

where $\tilde{J}^\alpha_{qq_1q_2}=[k_\beta\tilde{g}_q\tilde{I}_{q_2}+k_{2\beta}\tilde{g}_{q_2}\tilde{I}_q]\,\Delta^{\alpha\beta}_{\mathbf{k}_1}\tilde{F}_{q_1}$.

Substituting (4.3), expression (4.2) can be rewritten in the form

$$\varepsilon_T=S(d)\int_0^k\tilde{l}_{\mathbf{k}}k^{d-1}dk. \tag{4.4}$$

For power-law spatial spectra of temperature pulsations $I_{\mathbf{k}} = Ak^{-p}$ and of the velocity field $F_{\mathbf{k}} = C_i k^{-\alpha}$, $l_{\mathbf{k}}$ also assumes a scale invariant form $\tilde{l}_{\mathbf{k}} \sim k^{x-d}$. In this case, it follows from (4.4)

$$\varepsilon_T=S(d)k^d\;\;l_{\mathbf{k}}/x. \tag{4.5}$$

In this case, $x = 0$, Eq. (4.5) contains an indefinite expression 0/0. Expansion of the indefinite term leads to a relation between the flux and the derivative with respect to the index† [10]

*An additional integration with respect to ω removes the average over **v**.
†An analogous relation occurs in the theory of weak turbulence [13, 14].

$$\varepsilon_T = S(d)\,k^d \left.\frac{\partial \tilde{I}_{\mathbf{k}}}{\partial x}\right|_{x=0}. \tag{4.6}$$

In order to find the derivative, it is convenient to use the factorized expression for $\tilde{I}_{\mathbf{k}}$ in the form (3.4). Then, from (4.6) we obtain

$$\varepsilon_T = (\pi/2)\,S(d)\,k^d \int d^{(d)}\mathbf{k}_1 d^{(d)}\mathbf{k}_2 \delta^{(d)}(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)\,\Theta_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}\Delta^{\alpha\beta}_{\mathbf{k}_1}(k_\alpha \ln k + k_{2\alpha}\ln k_2)\,(k_\beta I_{\mathbf{k}_2} + k_{2\beta} I_{\mathbf{k}})\,F_{\mathbf{k}_1}, \tag{4.7}$$

where $\Theta_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} = \int\limits_0^\infty \frac{dt}{2\pi}\,h(k^s t)\,f(k^s t)\,r(k^s t)$ is the positive definite function [15].

Introducing the dimensionless variables as $k_1 = ku$, $k_2 = kv$, we rewrite (4.7) finally as follows:

$$\varepsilon_T = 2^{d-2} S(d)\,S(d-1)\,A C_i^{1/2} \int\!\!\int\limits_{\Delta} \frac{du\,dv}{u^2}\,\Theta_{1uv} v^{-p+1} u^{-\alpha+1}(v^p - 1)\ln v\,\Delta^{d-1}_{1uv}, \tag{4.8}$$

where $\Delta_{1uv} = (1/4)\sqrt{2(u^2v^2 + u^2 + v^2) - 1 - u^4 - v^4}$ is the area of the triangle with sides u, v, and 1, i.e., the quantity is essentially positive.

The integration in (4.8) is carried out over the region Δ, defined by the following inequalities: $|u - v| \leqslant 1 \leqslant u + v$. The direction of flow ε_T is given by the sign of the integral on the right side of (4.8). For sign $\varepsilon_T > 0$, the flow is directed toward increasing wave numbers, while for the opposite inequality, towards decreasing wave numbers. We note that the constant C_i in the Kolmogorov spectrum for the velocity field is related to the energy flux (or enstrophy for d = 2) by the relation $C_i \sim \varepsilon_i^{2/3}$. Taking this into account, we obtain

$$\operatorname{sign}\varepsilon_T = \operatorname{sign}\varepsilon_i \operatorname{sign}[(v^p - 1)\ln v] \equiv \operatorname{sign}\varepsilon_i \operatorname{sign} p.$$

For p = 0, the magnitude of the flux ε_T vanishes. This is related to the fact that this value of p corresponds to the thermodynamically equilibrium solution.

For nonequilibrium distributions p > 0 (both for d = 3 and d = 2), the flux of temperature inhomogeneities also always has a sign corresponding to the flux of the velocity field. As far as we know, the sign of ε_T has not been discussed previously in the literature.

5. Let us examine the behavior of the spectrum of temperature pulsations for large wave numbers. In this case, the shape of the spectrum $I_{\mathbf{k}}$ will depend greatly on the value of the Prandtl number $\Pr = \nu/\chi$. In what follows, we shall restrict our analysis to the case $\Pr \gg 1$, which is typical for most real fluids. For $\Pr \gg 1$, there exists a viscoconvective interval of wave numbers $\eta^{-1} \ll k \ll (\Pr)^{1/2}\eta^{-1} = (\varepsilon/\nu\chi^2)^{1/4}$, where the molecular viscosity already plays an important role, while the effect of molecular diffusion (heat conduction) is negligibly small. In this interval the kinetic energy spectrum decreases exponentially, while the spectrum of temperature pulsations with d = 3, according to Batchelor's semiempirical theory [16], varies according to the inverse first power law: $k^2 I_{\mathbf{k}} \sim k^{-1}$. It is difficult to obtain this result analytically, since scale invariance is already absent in the starting diagrammatic series (since the kinetic energy spectrum is not a power-law function) and it is formally necessary to include all terms in the equations for both I_q and g_q.

In the viscoheat-conducting interval, i.e., for $k \gg (\varepsilon/\nu\chi^2)^{1/4}$, both molecular viscosity and thermal conductivity are important and in this range of scales, the problem is considerably simplified [17, 18]. First of all, the transport interactions, which complicate the analysis in the inertial-convective interval, are unimportant here. Second, Green's function for the velocity field coincides with its value in the fluid at rest. As in [17, 18], it can be shown that the same results are also valid for the characteristics of the passive admixture. Thus, in the viscoheat conducting interval we obtain the following equation in the $\mathbf{k} - t$ representation for the spectrum of temperature pulsations from (2.2b):

$$I_{\mathbf{k}}(t - t') = \int d^{(d)}\mathbf{k}_1 d^{(d)}\mathbf{k}_2 \delta^{(d)}(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)\,k_\alpha k_\beta \Delta^{\alpha\beta}_{\mathbf{k}_1} \times \tag{5.1}$$

$$\times \int\limits_{-\infty}^{t} \frac{d\tau_1}{2\pi} \int\limits_{-\infty}^{t} \frac{d\tau_2}{2\pi}\, g^{(0)}_{\mathbf{k}}(t - \tau_1)\, g^{*(0)}_{\mathbf{k}}(t' - \tau_2)\, I_{\mathbf{k}_2}(\tau_1 - \tau_2)\, F_{\mathbf{k}_1}(\tau_1 - \tau_2) + \ldots$$

Carrying out the integration in (5.1) with respect to time (using the fact that $g^{(0)}_{k}(\tau) = 2\pi e^{-\chi k^2 \tau}$) and retaining for simplicity only the first term on the right side, for the single-time spectral functions $I_k \equiv I_k(0)$ we find

$$I_{\mathbf{k}} = \frac{1}{\chi^2 k^2} \int d^{(d)}\mathbf{k}_1 d^{(d)}\mathbf{k}_2 \delta^{(d)}(\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2) \left[1 - \frac{(\mathbf{k}\mathbf{k}_1)^2}{k^2 k_1^2}\right] I_{\mathbf{k}_2} F_{\mathbf{k}_1}. \quad (5.2)$$

The kinetic energy spectrum in the dissipation interval, as shown in [18], contains an exponential factor $F_k \sim e^{-k\eta}$, so that the region where $k_1 \ll k$ makes the main contribution to the integral in (5.2). Let us expand the integrand in (5.2) in a series in powers of k_1/k and including second-order terms

$$I(|\mathbf{k} + \mathbf{k}_1|) \simeq I_k + \left(k_1 \cos\theta + (1/2)\frac{k_1^2}{k}\sin^2\theta\right)\frac{dI_k}{dk} + (1/2)k_1^2\cos^2\theta\frac{d^2 I_k}{dk^2},$$

where θ is the angle between the vectors $\mathbf{k}$ and $\mathbf{k}_1$. Substituting this expression into (5.2) and integrating with respect to θ using the fact that $d^{(d)}\mathbf{k}_1 = S(d-1)k_1^{d-1}\sin^{d-2}\theta dk_1 d\theta$, we have

$$I_{\mathbf{k}} = \frac{\sqrt{\pi}\, S(d-1)\, \Gamma\left(\frac{d+1}{2}\right)}{(\chi k)^2\, \Gamma(d/2+1)} \int dk_1 k_1^{d-1} F_{k_1} \left\{ I_k + (1/2)\frac{k_1^2}{k}\frac{d+1}{d+2}\frac{dI_k}{dk} + (1/2)\frac{k_1^2}{d+2}\frac{d^2 I_k}{dk^2} \right\}. \quad (5.3)$$

It is convenient to rewrite Eq. (5.3) in the form

$$\frac{d^2 I_k}{dk^2} + \frac{d+1}{k}\frac{dI_k}{dk} + 2\left[\frac{2E_0\nu}{\varepsilon}(d+2) - k_2\frac{\chi^2\nu}{\varepsilon}d(d+2)\right] I_k = 0, \quad (5.4)$$

where $E_0 = \frac{d-1}{2} S(d) \int_0^\infty F_{k_1} k_1^{d-1} dk_1$ is the total kinetic energy per unit volume and $\varepsilon = (d-1)\nu S(d) \int_0^\infty F_{k_1} k^{d+1} dk_1$ is the average rate of dissipation of kinetic energy.

Equation (5.4) for $k \to \infty$ has the following asymptotic solution:

$$I_k \approx B \exp\left[-(1/2)\sqrt{\frac{2\chi^2\nu d(d+2)}{\varepsilon}}\, k^2\right].$$

The value of the parameter B can be found from the normalization condition $(d-1)\chi S(d) \int_0^\infty I_k \times k^{d+1} dk = \varepsilon_T$. This form of the exponential factor was found earlier by Batchelor [16] from semiempirical equations to within a constant $(1/2)\sqrt{2d(d+2)}$, which entered into the theory as an indefinite parameter. The analytical approach to the three-dimensional case gives the value $\sqrt{30}/2 \approx 2.74$ for this constant, which agrees well with the approximate experimental value of 2 [1].

In conclusion we emphasize that all of the equations presented above relate not only to temperature, but also to the concentration of an arbitrary passive admixture, for example, the moisture content or CO_2 concentration in the atmosphere, salinity of the ocean, electron density in the ionosphere, etc.

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SOME NEW PROBLEMS IN FILTRATION THEORY

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1. Invariant Γ-Integrals in Filtration Theory.

The stationary filtration of an incompressible liquid in a homogeneous isotropic porous medium is described by the following equations [1]:

$$\varphi_{,ii} = 0, \quad v_i = \varphi_{,i} \quad (i = 1, 2, 3), \quad \varphi = -(k/\rho g)p - kx_3, \tag{1.1}$$

where v_i are the components of the filtration velocity; p, pressure of the liquid; k, filtration coefficient; ρg, specific weight of the liquid; φ, velocity potential; x_1, x_2, x_3, rectangular Cartesian coordinates (the x_3 axis is directed opposite to the force of gravity).

Let Σ be an arbitrary closed surface in the porous medium under consideration. If within this surface there are no singular points, lines, or surfaces of the field, the following equations hold [2, 3]:

$$\int_\Sigma (v_i,\ v_i n_k - 2v_i n_i v_k)\, d\Sigma = 0; \tag{1.2}$$

$$\int_\Sigma [(v_i v_i)_{,l}\, n_k - 2(v_i,\ v_k)_{,l}\, n_i]\, d\Sigma = 0 \ \ (i, k, l = 1, 2, 3) \ldots, \tag{1.3}$$

where the n_k are the components of the unit normal vector to the surface Σ.

The proof of Eq. (1.2) follows from the transformations

$$\int_\Sigma (v_i v_i n_k - 2v_i v_k n_i)\, d\Sigma = \int_V [(v_i v_i)_{,k} - 2(v_i, v_k)_{,i}]\, dV = \int_V (2v_i v_{i,k} - 2v_i v_{k,i} - 2v_{i,i} v_k)\, dV = 0,$$

since, according to (1.1), $v_{i,k} = v_{k,i}$, $v_{i,i} = 0$ over the entire volume V within the surface Σ. The proof of (1.3) and other such equations is analogous.

If within the surface Σ there are singular points, lines, or surfaces of the field, then obviously the left side of Eq. (1.2) will remain unchanged under any deformations of Σ which do not affect the singularities of the field.

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